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Volumetric colon wall unfolding using harmonic differentials

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A B S T R A C T

Volumetric colon wall unfolding is a novel method for virtual colon analysis and visualization with valuable applications in virtual colonoscopy (VC) and computer-aided detection (CAD) systems. A volumetrically unfolded colon enables doctors to visualize the entire colon structure without occlusions due to haustral folds, and is critical for performing efficient and accurate texture analysis on the volumetric colon wall. Though conventional colon surface flattening has been employed for these uses, volumetric colon unfolding offers the advantages of providing the needed quantities of information with needed accuracy. This work presents an efficient and effective volumetric colon unfolding method based on harmonic differentials. The colon volumes are reconstructed from CT images and are represented as tetrahedral meshes. Three harmonic 1-forms, which are linearly independent everywhere, are computed on the tetrahedral mesh. Through integration of the harmonic 1-forms, the colon volume is mapped periodically to a canonical cuboid. The method presented is automatic, simple, and practical. Experimental results are reported to show the performance of the algorithm on real medical datasets. Though applied here specifically to the colon, the method is general and can be generalized for other volumes.

1. Introduction

Colorectal cancer is the third most incident cancer worldwide [16]. It is recommended that the elderly and at-risk are regularly screened for polyps, the precursors of cancer, located on the colon wall. This has traditionally been accomplished by performing an optical colonoscopy (OC), where an endoscope is inserted into the colon through the rectum. Due to the inherent discomfort in this procedure and its bowel preparation, and the risk associated with this invasive procedure, virtual colonoscopy (VC) systems have been developed [7]. In VC, computed tomographic (CT) images of the patient’s abdomen are used to reconstruct a virtual colonic model. Doctors are then able to navigate through this model using a virtual fly-through visualization system and search for polyps. Recent studies have shown such systems to be as effective as or better than traditional OC methods [10].

VC systems, however, share some inherent disadvantages with OC. Inspection of the entire colonic wall can be time consuming due to the length of the colon. Worse, however, is that due to the haustral folds present on the colon walls, many areas are hidden in the standard view, and any polyps in these areas are therefore missed, leading to incomplete examinations. One solution to overcome this limitation is to map the entire surface of the colon to a plane, such that there are no occlusions. Conformal flattening is an advantageous method for this since the resulting mapping is angle preserving and the global distortion is minimized [8]. The property of being angle preserving is especially important, as this means that shapes will be preserved; thus the shape of a polyp will still be readily identifiable to a user (doctor) trained in identifying polyps in 3D. The flattened 2D mesh can be rendered using direct volume rendering with GPU acceleration. In addition to visualization, conformal flattening of the colon surface has been applied for computer-aided detection (CAD) of colonic polyps using electronic biopsy [9] and for registration of supine and prone colon surfaces [31].

The anatomy of the colon wall, however, is intrinsically volumetric. Recent work has shown that it is possible to detect and extract the outer colon wall from the CT images [24]. Utilizing this volumetric data, we propose a method for volumetric colon unfolding, which offers more complete and accurate information. For visualization, rendering the volumetric colon wall can offer better results due to the automatic presence of the volume rendering view vector. That is, given the inner and outer colon surfaces flattened to two parallel planes (i.e., the front and back faces of the cuboid), the view vector can be immediately defined for any point on the surface. Using only a single surface, this view vector must be calculated in some manner (e.g., from the nearest point on the centerline, using the surface normal, etc.), which can lead to slightly inaccurate results. For virtual biopsy in VC and CAD, using the volumetric colon wall in conjunction with volume rendering integration ensures not only that the view ray is correct, but also ensures that the accumulation of the discrete
volume rendering integral is finished when the colon outer wall is reached. For supine-prone registration, the volumetric colon presents additional information that can be used to arrive at a more accurate registration result.

Our method is based on three harmonic 1-forms, which can be treated as three vector fields, such that both the divergence and circulation of them are zero. The volume is unfolded to a canonical cuboid by integrating over these three harmonic 1-forms. The proposed method has the following advantages:

1. **Rigorous and theoretically solid**: The algorithm is based on Hodge theory.
2. **Automatic**: It also allows for human intervention.
3. **Efficient**: It is to solve large sparse linear systems.
4. **Shape preserving as much as possible**: The global distortion from colon volume to parametric cuboid, measured by harmonic energy, is minimized.
5. **General**: It can be generalized to deal with the volumes with more complicated topologies.

Combined with the direct volume rendering, the 2D colon image provides an efficient way to augment VC systems. The unfolding of a 3D volumetric colon is a fundamental step in the implementation of further virtual colon processing algorithms, such as enhanced CAD and registration techniques, as well as new visualization possibilities.

## 2. Related work

Visualization methods have been proposed to assist the medical doctor in ensuring a complete inspection during VC, such as the unfolded cube projection method [25]. A popular method for showing the entire colon at once is that of virtual dissection. Early work focused on the straightforward method of using cross sections through the colon for each point by straightening the path of the colon and map the local regions to different 2D frames. However, it does not provide a complete overview of the colon. They presented a further two step technique to deal with the double appearance of polyps and nonuniform view of the colon. They introduced a method based on harmonic 1-forms [8], which flattens the colon surface to a rectangle. More recently, Qiu et al. proposed a method for conformal flattening based on discrete Ricci flow [17] to deform the Riemannian metric using the heat diffusion equation of curvature flow with a target Gaussian curvature of zero everywhere.

Many colon CAD systems use the geometry of the colon surface to perform shape analysis for areas that have the appearance of a polyp (bulbous convex structures on colon walls which protrude into the colon lumen). These methods to characterize the curvature measures of the colon wall in CAD systems include shape index and curvedness [30,13], shape description based on global curvature [29], the intersection of normals [15], and sphere fitting [21,11]. CAD of the colon polyps has also been implemented using the conformally flattened colon surface rendered by direct volume rendering with a transluscent transfer function [9].

In recent years, the discrete one-form has been applied to many applications in computer graphics and visualization, such as surface parameterization [5,3], quad mesh design [22], vector field design [23,12], and so on. We refer readers to [4] for a more thorough list of applications and references.

## 3. Theoretic background

In practice, all the shapes are approximated by tetrahedral meshes, namely, simplicial complexes. In the following discussion, all of the concepts are defined on meshes directly. More details can be found in [4].

Suppose $M$ is a simplicial complex. We use $[v_0,v_1,\ldots,v_n]$ to represent an $n$-simplex. For example, vertex, edge, face, and tetrahedron are simplices.

### 3.1. Closed and exact forms

A $k$-chain $\gamma_k$ is a linear combination of all $k$-simplices in $M$, $\gamma_k = \sum \sigma_i^k$. All $k$-chains form a linear space, called the $k$-dimensional chain space $C_k$. The $k$-dimensional boundary operator $\partial_k: C_k \rightarrow C_{k-1}$ is a linear operator, defined as taking the boundary of a $k$-chain,

$$\partial_k \left( \sum \sigma_i^k \right) = \sum \partial_k \sigma_i^k,$$

where $\sigma_i^k$ goes through all $k$-simplices in $M$. On each simplex,

$$\partial_k [v_0,v_1,\ldots,v_k] = \sum_{i=0}^{k} (-1)^i [v_0,\ldots,\hat{v_i},\ldots,v_k],$$

where $[v_0,\ldots,\hat{v_i},\ldots,v_k]$ represents the $(k-1)$-simplex with vertices from $v_0$ to $v_k$ except $v_i$.

A $k$-form is a linear function defined on the chain space, $\omega_k: C_k \rightarrow \mathbb{R}$,

$$\omega_k(\gamma_k) = \sum_{i=0}^{k} \omega_k(\sigma_i^k).$$

Sometimes, the action of $\omega_k$ on $\gamma_k$ is also denoted as $\langle \omega_k,\gamma_k \rangle$. All $k$-forms form a linear space, the so-called $k$-dimensional co-chain space $C^k$. The $k$-dimensional co-boundary operator $d_k: C^k \rightarrow C^{k+1}$ is a linear operator, defined as the dual operator of the boundary operator $\partial_{k+1}$,

$$\langle d_k \omega, \gamma_{k+1} \rangle = \langle \omega, \partial_{k+1} \gamma_{k+1} \rangle.$$  \hfill (1)

The above equation is called the Stokes formula.

Suppose $\omega$ is a $k$-form, then $\omega$ is an exact form, if there exists a $(k-1)$-form $\tau$, such that $\omega = d_{k-1} \tau$; $\omega$ is a closed form, if $\partial_k \omega = 0$. It can be verified easily that all exact forms are closed. Let $\omega_1$ and $\omega_2$ be closed k-forms; if they differ by an exact k-form, $\omega_1 - \omega_2 = d\xi$, then they are cohomologous. All the cohomologous
classes of closed k-forms form the k-dimensional cohomology group of M. Symbolically,
\[ H^k(M, R) = \frac{\ker d_k}{\text{img } d_{k+1}}, \]
where \( \ker d_k \) represents the kernel of \( d_k \), which is the linear space of all closed k-forms; \( \text{img } d_{k+1} \) represents the image of \( d_{k+1} \), which is the linear space of all exact k-forms.

3.2. Harmonic forms

Suppose \( M \) is embedded in \( \mathbb{R}^3 \), \( f : C_0 \to \mathbb{R} \) is a 0-form on \( M \), namely, a function defined on vertices. Then \( f \) can be extended to a piecewise linear function on \( M \). The harmonic energy of \( f \) is defined as
\[ E(f) = \int_M |\nabla f|^2 \, dv, \]
where \( \nabla f \) is the gradient of \( f \), \( dv \) is the volume element on \( M \). (\( f \) is piecewise linear, therefore \( \nabla f \) is well-defined on the interior points of all simplices, the singular sets of \( \nabla f \) are of zero measure, and the integration is well defined.) A harmonic function is a critical point of the harmonic energy.

Let \( [v_i, v_j] \) be an edge in \( M \) and let \( \theta_{ij} \) represent the dihedral angle on edge \( [v_i, v_j] \) in the tetrahedron \([v_i, v_j, v_k, v_l] \). The edge weight on \([v_i, v_j] \) is defined as
\[ w_{ij} = \sum_{kl} \cot \theta_{kl}. \]
Let \( f \) be a 0-form, then the discrete Laplacian of \( f \) is a 0-form, defined as
\[ \Delta f(v_i) = \sum_j w_{ij} (f(v_j) - f(v_i)) = \sum_j w_{ij} \omega([v_i, v_j]), \]
where \( \omega = df \). It can be easily verified that harmonic functions have zero Laplacians.

Suppose \( \omega \) is a closed 1-form, \( \omega \) is harmonic if locally it is the discrete derivative of a harmonic function. Namely,
\[ \forall i, \sum_j w_{ij} \omega([v_i, v_j]) = 0. \]

According to the following Hodge theorem [4], there exists a unique harmonic 1-form in each cohomologous class in \( H^1(M, R) \).

**Theorem 1** (Hodge). Each cohomologous class has a unique harmonic differential form.

Our work focuses on finding three different harmonic 1-forms on \( M \), whose existences are guaranteed by Hodge theory.

4. Algorithm

The volumetric colon is a topological cylinder with thickness. We first cut the volumetric cylinder to a volumetric ball (a fundamental domain) along a special surface (meridian); then compute three exact and non-exact harmonic 1-forms; finally, we integrate harmonic 1-forms on the fundamental domain to map the volume to a canonical cuboid. We use the volume in Fig. 1 to explain the computational details of the volumetric unfolding algorithm.

4.1. Compute meridian

Suppose \( M \) is a 3-manifold and the boundary of \( M \) consists of a single surface, then \( M \) is called a handle body. Suppose the boundary \( \partial M \) is of genus \( g \), then there exists \( g \) topological disks \( \{D_1, D_2, \ldots, D_g\} \), such that \( M - (\bigcup_i D_i) \) is a topological ball. Such topological disks are called meridians of \( M \) (see Fig. 2). Computing meridians is crucial for topological denoising and unfolding. The algorithm for computing meridians consists of two stages:

**Stage 1**: Find a cut graph \( G \) of \( M \), which includes the meridians, using the following steps:

1. Construct the dual graph of the tetrahedral mesh \( M \), denoted as \( G \).
2. Compute a minimal spanning tree \( T \) of \( G \).
3. Suppose \( G - T = \{e_1, e_2, \ldots, e_k\} \), each \( e_i \) corresponds to a face \( f_i \in M \), then \( G' = \bigcap_{i=1}^k f_i \).

**Stage 2**: Retract the subcomplex \( G' \) to the meridians, using the following steps:

1. For any simplex \( \sigma \subset G' \) with dimension \( k \), compute its valence, which is the number of \( k+1 \) dimensional simplices in \( G' \) adjacent to \( \sigma \).
2. Find all valence 1 \( k \)-simplices in \( G' \), remove them and their unique \( k+1 \)-simplex neighbor from \( G' \), \( k = 0, 1, 2 \).
3. Update the valence of the simplices in \( G' \).
4. Repeat steps 2 and 3, until there is no valence 1 simplex in \( G' \).
5. The connected components of \( G' \) are meridians \( \{D_1, D_2, \ldots, D_g\} \).

Along the meridians, we cut the volume \( M \) open to obtain a topological ball, denoted as \( \bar{M} \). Each tetrahedron \([v_i, v_j, v_k, v_l] \) has four corners; the corner at \( v_i \) is denoted as \( n_{ij} \). For two adjacent corners \( n_{ij} \) and \( n_{kl} \) at the vertex \( v_i \), we call face \([v_i, v_j, v_k] \) their common face. A wedge is a collection of corners at a vertex, \( w = \{n_{11}, n_{12}, \ldots, n_{1m}\} \), such that \( n_{ij} \) and \( n_{ik} \) are adjacent, and their common face is not in the cut graph \( G' \), \( k = 1, 2, \ldots, n-1 \). If a vertex is not on \( G' \), then all its corners belong to the same wedge.
where $D$ is a topological ball, thus we call it the fundamental domain of $M$.

4.2. Compute exact harmonic 1-forms

The boundary of the colon volume $M$ is a torus. As shown in Fig. 3, the boundary surface is segmented into four regions. Two annuli, $\Omega_1, \Omega_2$, are illustrated by the green and yellow regions on the surface; two cylinders, $\Omega_0, \Omega_2$, are illustrated by the red and blue regions. We then compute harmonic functions by solving the Dirichlet problem on $M$,

$$
\begin{align*}
\begin{cases}
  f_0|_{\partial D} = 1, \\
  f_0|_{\partial D} = 0, \\
  \Delta f_0(v_i) = 0, \quad v_i \notin \partial D,
\end{cases}
\end{align*}
$$

where $\Delta f(v_i) = \sum w_j (f_j(v_i) - f_i(v_j))$, $l = 1, 2$. Figs. 4 and 5 show the level sets of the harmonic functions $f_0$ and $f_1$, respectively. The gradients $\omega_0 = df_0$ and $\omega_1 = df_1$ are exact harmonic 1-forms.

Fig. 3. Boundary surface segmentation. (a) View 1, (b) view 2. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

4.3. Compute non-exact harmonic 1-form

Suppose the meridian $D$ and the fundamental domain $M$ has been computed. Each vertex $w_i$ in $M$ is a wedge in $M$, and therefore has a unique corresponding vertex in $M$. We denote the vertex mapping as $\phi : M \to M$.

Consider that the preimage of the meridian $D$, $\phi^{-1}(D)$, has two connected components $\tilde{D}^+, \tilde{D}^-$ on the boundary of $M$. Let $h$ be a 0-form on $M$, such that $h|_{\partial M} = 1$ and $h|_{\partial M} = 0$; $h$ is arbitrary on other vertices. Then we define a closed 1-form $\omega$ on $M$ such that

$$
\omega([\phi(w_i), \phi(w_j)]) = dh([w_i, w_j]).
$$

Note that each edge $[w_i^+, w_j^+]$ on $\bar{D}^+$ corresponds to a unique edge $[w_i^-, w_j^-]$ on $\bar{D}^-$. By construction, $dh([w_i^+, w_j^+]) = df([w_i^-, w_j^-])$. Therefore, $\omega$ is well defined. Furthermore, $\omega$ is closed. Let $\omega_2$ be the unique harmonic 1-form on $M$ homologous to $\omega$. Then $\omega_2$ and $\omega$ differ by an exact form $dg$, where $g$ is a 0-form. By the harmonicity condition,

$$
\sum_{j} w_j(g(v_j) - g(v_i) + \alpha([v_i, v_j])) = 0, \quad \forall v_i.
$$

g can be uniquely determined by solving the linear system. Therefore, harmonic 1-form $\omega_2$ can be constructed. The level sets are shown in Fig. 6.

4.4. Compute integration of harmonic 1-forms

After computing the harmonic 1-forms $\omega_0, \omega_1, \omega_2$, we convert them to three piecewise constant vector fields $e_0, e_1, e_2$. Let $\omega$ be a closed 1-form, on each tetrahedron $[v_i, v_j, v_k, v_s]$, we find a vector $e$, such that

$$
\omega([v_i, v_j]) = \langle v_j - v_i, e \rangle.
$$

We find three constants $c_0, c_1, c_2$, such that the vector fields $c_0 e_0, c_1 e_1, c_2 e_2$ locally form a frame in $\mathbb{R}^3$; the frame is as close to
an orthogonal frame as possible. Therefore, we minimize the following energy:

\[
E(c_0, c_1, c_2) = \sum_{i,j,k} \left| c_i \mathbf{e}_i \times \mathbf{e}_j - c_k \mathbf{e}_k \right|^2 dv,
\]

where \(i, j, k\) traverse all the cyclic permutations of \(0, 1, 2\). In practice, we can assume \(c_0\) to be 1. For convenience, we still use \(\omega_0, \omega_1, \omega_2\) to denote the scaled harmonic 1-forms.

Given a closed 1-form \(\omega\) on \(M\), we can define the corresponding 1-form \(\phi^* \omega\) on \(\tilde{M}\) as the following:

\[
\phi^* \omega([w_i, w_j]) = \omega([w_i, w_j]).
\]

Then \(\phi^* \omega\) is a closed 1-form on \(\tilde{M}\). We choose an arbitrary vertex \(v_0 \in \tilde{M}\) as base vertex. For any vertex \(v \in \tilde{M}\), we find an arbitrary path \(\gamma = v_0, v_1, \ldots, v_n, v_n = v\), connecting \(v_0\) and \(v\). Because \(\phi^* \omega\) is closed, and \(\tilde{M}\) is a topological ball, then the integration value \(\langle \phi^* \omega, \gamma \rangle\) is independent of the choice of the path by Stokes formula (1). We denote the mapping as \(\psi : \tilde{M} \to \mathbb{R}^3, \psi(v_k) = \phi(v_k) + (\phi^* \omega_0 [v_k, v_{k+1}]), \phi^* \omega_1 [v_k, v_{k+1}], \phi^* \omega_2 [v_k, v_{k+1}]), k = 0, n-1\). As shown in Fig. 7, it maps the volumetric cylinder to cuboid periodically. \(\psi(D^-)\) and \(\psi(D^+)\) differ by a translation. By cutting and merging, the parametric domain becomes a canonical cuboid, essentially, equivalent to a canonical cylinder. The radius of the canonical cylinder is fully determined by the Riemannian metric of the original volume.

5. Experimental results

We have implemented our volumetric colon unfolding and rendering algorithm in C/C++ with GPU acceleration. We run all the experiments on a dual-processor 3.60 GHz Xeon PC running 32 bit windows XP, with 3.00G RAM and NVIDIA Geforce 8800 graphics board.
5.1. Preprocessing

We perform the following tasks to extract the colon volume from the CT dataset. First, a segmentation algorithm based on level sets [24] is applied, and a binary mask is generated, which labels the voxels belonging to the colon wall. This algorithm is applied to a colon dataset which has been previously cleansed with the partial volume effect taken into account. With the segmentation of the colon wall, we then use VTK to extract a topologically simple tetrahedral mesh.

The large resolution of colon CT dataset (512 x 512 x 451) requires large memory for storage and the subsequent volumetric unfolding. These requirements are higher than our current system specifications. It requires us to split the colon volume into sections for separate processing. These sections can be sewn back together after the volumetric computation.

5.2. Volumetric unfolding

Our experiments show that the numerical stability is strongly influenced by the triangulation. Comparing to regular grids tessellation, volumetric Delaunay triangulation [20] gives better performance, in terms of both stability and convergence rate. On the other hand, in order to represent the unstructured tetrahedral mesh, we need to use the half-face data structure, which has a high storage cost. We use the MatLab C++ library for solving the large sparse linear systems.

We conducted our experiments on a 32-bit platform; therefore due to the memory limitation, we can only test our algorithms on part of the colon volumes. Fig. 8 shows some experimental results. The statistics are shown in Table 1. The computing of the harmonic 1-form $\omega_2$ is the most time consuming part in the unfolding process. For large data sets, such as the third model with more than 380k tetrahedra, the total processing time is less than 2 min.

5.3. Volumetric rendering

Our primary purpose for volumetric unfolding of the 3D colon wall is to enable further processing and algorithm development which takes into account the entire colon structure rather than just the mucosal wall. However, these unfolded 3D colon segments can also be used directly for visualization. For this, we render the unfolded volumetric mesh using the cell projection method [19], which is accelerated on the GPU [18]. For the volume rendering, we use a translucent transfer function to accumulate the density values through the entire colon wall, resulting in an electronic biopsy image similar to previous CAD work [9].

Fig. 8 shows examples of sections of the volumetrically unfolded colon rendered with the translucent transfer function described above. The transfer function maps lower density values to blue and green, while higher density values are mapped to red. The first image, from the end of the colon which generally contains higher density values, shows the reddish color throughout, while the other images show mostly blue or green. Polyps, which would appear as high density in either red or yellow, were not detected in the VC examination of this colon. These rendered images are created by looking down at the unfolded colon inner surface. Rendering from the side, while not necessarily of help for a CAD system, does present the user with a better feel for the density anatomy of the colon. Again showing the end of the colon, it can be seen here how the inner surface contains the low density soft tissue, while the outer colon is the portion that contributes the high density values.

5.4. Discussion

Generality: Because of the special geometry of colon volumes, no zero points have been detected in our experiments. For general geometry, it will be challenging to locate and remove zero points. By Hodge theory, our algorithm can be generalized to handle more complicated shapes. Especially, this method can be applied for other organs with different shapes for abnormality detection and shape analysis in the medical imaging field. The current implementation uses half-face data structure requiring high memory storage. It is practical to design a more compact data structure and use a 64-bit platform to handle the entire colon volume.

Utility: The unfolding of a 3D volumetric colon wall to a canonical cuboid is a fundamental step towards the realization of further computational processing algorithms for colon volumes. This provides an efficient and useful approach to developing new VC interfaces and CAD systems. The various medical applications can be created by using the volumetric unfolded colon, which preserves shapes as much as possible. In addition to augmenting the VC interface, it can be used in CAD work for colonic polyp detection. The texture and shape analysis methods can also be implemented accordingly. With the additional information contained in the colon volume compared to only the colon surface, new methods have the possibility of providing superior results compared to existing techniques. We are also exploring the possibility of volumetric colon registration and fusion between supine and prone scans of the same patient. This volumetric framework can be applied for shape analysis of different organ datasets. It is immediately applicable to other tube-like structures, such as blood vessels, nasal passages, and the urethra.

6. Conclusion

This paper introduces an automatic and practical algorithm for volumetric colon wall unfolding. The method uses harmonic 1-forms, whose existences are guaranteed by Hodge theory. By solving sparse linear systems, three harmonic 1-forms can be computed, and the unfolding is then obtained by integrating the 1-forms. The experimental results tested on real colon datasets show the efficiency and efficacy. We have shown the volume rendered results of the unfolded colons, which illustrate that the method can successfully be used to visualize the density of the colon volume.

The unfolding of a 3D colon wall to a canonical cuboid is a fundamental step to allow for further computational colon volume processing algorithms, and we are investigating its use in three applications. For creating a 2D unfolded overview of the entire colon volume, rendering from the 3D unfolded data allows for view rays into the colon wall to be determined automatically. For CAD of colonic polyps, performing analysis in 3D cuboid gives significantly more information than a 2D image, while being in a canonical shape still allows for efficient processing. Likewise, registration of two colon volumes can also be enhanced by this additional information while maintaining the computational efficiency.
Acknowledgments

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